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INTEGRAL REPRESENTATION FORMULAE IN HERMITEAN CLIFFORD ANALYSIS

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Abstract. *Euclidean Clifford analysis is a higher dimensional function theory offering a refinement of classical harmonic analysis. The theory is centered around the concept of monogenic functions, i.e. null solutions of a first order vector valued rotation invariant differential operator called the Dirac operator, which factorizes the Laplacian. More recently, Hermitean Clifford analysis has emerged as a new and successful branch of Clifford analysis, offering yet a refinement of the Euclidean case; it focusses on the simultaneous null solutions, called Hermitean (or h–) monogenic functions, of two Hermitean Dirac operators which are invariant under the action of the unitary group. In Euclidean Clifford analysis, the Clifford–Cauchy integral formula has proven to be a corner stone of the function theory, as is the case for the traditional Cauchy formula for holomorphic functions in the complex plane. Previously, a Hermitean Clifford–Cauchy integral formula has been established by means of a matrix approach. This formula reduces to the traditional Martinelli–Bochner formula for holomorphic functions of several complex variables when taking functions with values in an appropriate part of complex spinor space. This means that the theory of Hermitean monogenic functions should encompass also other results of several variable complex analysis as special cases. At present we will elaborate further on the obtained results and refine them, considering fundamental solutions, Borel–Pompeiu representations and the Teoderescu inversion, each of them being developed at different levels, including the global level, handling vector variables, vector differential operators and the Clifford geometric product as well as the blade level were variables and differential operators act by means of the dot and wedge products. A rich world of results reveals itself, indeed including well–known formulae from the theory of several complex variables.*

1 INTRODUCTION

The Cauchy integral formula for holomorphic functions in the complex plane may be generalized to the case of several complex variables in two ways: either one takes a holomorphic kernel and an integral over the distinguished boundary $\partial_0 \widetilde{D} = \prod_{j=1}^n \partial \widetilde{D}_j$ of a polydisk $\widetilde{D} = \prod_{j=1}^n \widetilde{D}_j$ in $\mathbb{C}^n$, leading to

$$f(z_1, \ldots, z_n) \;=\; \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \wedge \cdots \wedge d\xi_n \,, \quad z_j \in \overset{\circ}{\widetilde{D}}_j \tag{1}$$

or one takes an integral over the (piecewise) smooth boundary ∂D of a bounded domain D in $\mathbb{C}^n$ in combination with the Martinelli–Bochner kernel, see e.g. [15], which is not holomorphic anymore but still harmonic, resulting into

$$f(z) \;=\; \int_{\partial D} f(\xi)\, U(\xi, z) \,, \quad z \in \overset{\circ}{D} \tag{2}$$

with

$$U(\xi, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\xi_j^c - z_j^c}{|\xi - z|^{2n}} d\xi_1^c \wedge \cdots \wedge d\xi_{j-1}^c \wedge d\xi_{j+1}^c \wedge \cdots \wedge d\xi_n^c \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$$

where $\cdot^c$ denotes the complex conjugate. The history of formula (2), obtained independently by Martinelli and Bochner, has been described in detail in [14]. It reduces to the traditional Cauchy integral formula when $n = 1$; for $n > 1$, it is related to the double layer potential, while at the same time, it establishes a connection between harmonic and holomorphic functions.

A third alternative for a generalization of the Cauchy integral formula is offered by Clifford analysis, where functions defined in Euclidean space $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and taking values in a Clifford algebra are considered. One focusses on so–called monogenic functions, i.e. null solutions of the elliptic Dirac operator $\partial_{\underline{X}}$ factorizing the Laplace operator: $\partial_{\underline{X}}^2 = -\Delta_{2n}$. As the Dirac operator is rotation invariant, the name Euclidean Clifford analysis is used nowadays to refer to this setting. Standard references are [7, 11, 13, 12]. In this framework the kernel appearing in the Clifford–Cauchy formula is monogenic, up to a pointwise singularity, while the integral is taken over the complete boundary:

$$f(\underline{X}) = \int_{\partial D} E(\underline{\Xi} - \underline{X})\, d\sigma_{\underline{\Xi}}\, f(\underline{\Xi}) \,, \quad \underline{X} \in \overset{\circ}{D}$$

with

$$E(\underline{\Xi} - \underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{\Xi}} - \overline{\underline{X}}}{|\underline{\Xi} - \underline{X}|^{2n}}$$

a_{2n} being the area of the unit sphere S^{2n-1} in $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $\bar{\cdot}$ denoting the Clifford conjugation and $d\sigma_{\underline{\Xi}}$ being a Clifford algebra valued differential form of order $(2n-1)$. This Clifford-Cauchy integral formula is a corner stone in the development of the function theory.

In a series of recent papers, so–called Hermitean Clifford analysis has emerged as yet a refinement of the Euclidean case; it focusses on the simultaneous null solutions of the complex Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ which decompose the Laplace operator in the sense that $4(\partial_{\underline{Z}} \partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger} \partial_{\underline{Z}}) = \Delta_{2n}$ and which are invariant under the action of the special unitary group.

The study of complex Dirac operators was initiated in [17, 16, 18]; a systematic development of the associated function theory, including the invariance properties with respect to the underlying Lie groups and Lie algebras, is still in full progress, see e.g. [9, 8, 3, 4, 1, 2].

A Cauchy integral formula for Hermitean monogenic functions taking values in the complex Clifford algebra $\mathbb{C}_{2n}$ was essential in the further development of this function theory. A first result in this direction was obtained in [19], however for functions which are null solutions of only one of the Hermitean Dirac operators and moreover presenting a "fake" – as termed by the authors – Cauchy kernel, failing to be monogenic. In [6] a Cauchy integral formula for Hermitean monogenic functions has been established. However, from the start it was clear that the desired formula could not have the traditional form of (1) or (2). Indeed, it is known (see [4]) that in the special case where the functions considered do not take their values in the whole Clifford algebra $\mathbb{C}_{2n}$, but in the n-homogeneous part $\mathbb{S}_n$ of the complex spinor space $\mathbb{S} = \mathbb{C}_{2n}I \cong \mathbb{C}_n I$, I being a self-adjoint primitive idempotent, Hermitean monogenicity is equivalent with holomorphy in the underlying complex variables. It turned out that a matrix approach was the key to obtain the desired result. Moreover and as could be expected, the obtained Hermitean Cauchy integral formula reduces to the traditional Martinelli–Bochner formula (2) in the special case of functions taking values in a particular part of complex spinor space. This also means that the theory of Hermitean monogenic functions not only refines Euclidean Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, even encompassing some of its results.

2 PRELIMINARIES OF HERMITEAN CLIFFORD ANALYSIS

The real Clifford algebra $\mathbb{R}_{0,m}$ is constructed over the vector space $\mathbb{R}^{0,m}$ endowed with a non–degenerate quadratic form of signature $(0, m)$ and generated by the orthonormal basis $(e_1, \ldots, e_m)$. The non–commutative geometric multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_j e_k + e_k e_j = -2\delta_{jk} \ , \ \ j, k = 1, \ldots, m \tag{3}$$

As a basis for $\mathbb{R}_{0,m}$ one takes for any set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$ the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \cdots < j_h \leq m$, together with $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so–called k–vector part of a ($k = 0, 1, \ldots, m$). Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(X_1, \ldots, X_m)$ with the Clifford vector

$$\underline{X} = \sum_{j=1}^{m} e_j \, X_j$$

It holds that $\underline{X}^2 = - < \underline{X}, \underline{X} > = -|\underline{X}|^2$. The Fischer dual of $\underline{X}$ is the vector valued first order differential operator

$$\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \, \partial_{X_j}$$

called Dirac operator, and underlying the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^{0,m}$ and taking values in $\mathbb{R}_{0,m}$ is called (left) monogenic in Ω if $\partial_{\underline{X}}[f] = 0$ in Ω. As the Dirac operator factorizes the Laplacian: $\Delta_m = -\partial_{\underline{X}}^2$, monogenicity can be regarded as a refinement of harmonicity. We refer to this setting as the Euclidean case, since the fundamental group leaving the Dirac operator $\partial_{\underline{X}}$ invariant is the special orthogonal group SO$(m;\mathbb{R})$, which is doubly covered by the Spin(m) group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is also called rotation invariant.

When allowing for complex constants and moreover taking the dimension to be even: $m = 2n$, the same generators $(e_1, \ldots, e_{2n})$, still satisfying the multiplication rules (3), produce the complex Clifford algebra $\mathbb{C}_{2n}$, which is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\,\mathbb{R}_{0,2n}$. Any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may thus be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, an observation leading to the definition of the Hermitean conjugation $\lambda^\dagger = (a+ib)^\dagger = \overline{a} - i\overline{b}$, where the bar notation stands for the usual Clifford conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti–involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, 2n$. This Hermitean conjugation also leads to a Hermitean inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}$.

This is the framework for so–called Hermitean Clifford analysis, a refinement of Euclidean Clifford analysis. An elegant way of introducing this setting consists in considering a so–called complex structure, i.e. a specific SO$(2n;\mathbb{R})$–element J for which $J^2 = -\mathbf{1}$ (see [3, 4]). Here, J is chosen to act upon the generators $e_1, \ldots, e_{2n}$ of the Clifford algebra as

$$J[e_j] = -e_{n+j} \qquad \text{and} \qquad J[e_{n+j}] = e_j, \qquad j = 1, \ldots, n$$

With J one may associate two projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ which produce the main objects of the Hermitean setting by acting upon the corresponding objects in the Euclidean framework. First of all, the so–called Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are obtained through the action of $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal basis elements e_j:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j}), \quad j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j}), \quad j = 1, \ldots, n \end{aligned}$$

These Witt basis elements satisfy the Grassmann identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0 \ , \ \ j, k = 1, \ldots, n$$

and the duality identities

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk} \ , \ \ j, k = 1, \ldots, n$$

A vector $\underline{X} = (X_1, \ldots, X_{2n})$ in $\mathbb{R}^{0,2n}$ is now denoted by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and identified with the Clifford vector $\underline{X} = \sum_{j=1}^n (e_j\, x_j + e_{n+j}\, y_j)$; the action of the complex structure J on $\underline{X}$ yields

$$\underline{X}| = J[\underline{X}] = \sum_{j=1}^n (e_j\, y_j - e_{n+j}\, x_j)$$

The Clifford vectors $\underline{X}$ and $\underline{X}|$ anti–commute, since the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal w.r.t. the standard Euclidean scalar product. The actions of the projection operators on the Clifford vector $\underline{X}$ then produce the conjugate Hermitean Clifford variables $\underline{Z}$ and $\underline{Z}^\dagger$:

$$\begin{aligned} \underline{Z} &= \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] = \frac{1}{2}(\underline{X} + i\,\underline{X}|) \\ \underline{Z}^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] = -\frac{1}{2}(\underline{X} - i\,\underline{X}|) \end{aligned}$$

which may also be rewritten in terms of the Witt basis elements as

$$\underline{Z} \;=\; \sum_{j=1}^{n} \mathfrak{f}_j \, z_j \quad \text{and} \quad \underline{Z}^\dagger \;=\; (\underline{Z})^\dagger \;=\; \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, z_j^c$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, \ldots, n$. Finally, the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are obtained from the Euclidean Dirac operator $\partial_{\underline{X}}$:

$$\begin{aligned}
\partial_{\underline{Z}^\dagger} &= \frac{1}{4}(\mathbf{1} + iJ)[\partial_{\underline{X}}] &&= \frac{1}{4}(\partial_{\underline{X}} + i\,\partial_{\underline{X}|}) \\
\partial_{\underline{Z}} &= -\frac{1}{4}(\mathbf{1} - iJ)[\partial_{\underline{X}}] &&= -\frac{1}{4}(\partial_{\underline{X}} - i\,\partial_{\underline{X}|})
\end{aligned}$$

where also the so–called twisted Dirac operator arises:

$$\partial_{\underline{X}|} \;=\; J[\partial_{\underline{X}}] \;=\; \sum_{j=1}^{n} (e_j \, \partial_{y_j} - e_{n+j} \, \partial_{x_j})$$

As for $\partial_{\underline{X}}$, a notion of monogenicity may be associated in a natural way to $\partial_{\underline{X}|}$ as well. Passing to the Witt basis, the Hermitean Dirac operators are expressed as

$$\partial_{\underline{Z}} \;=\; \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \partial_{z_j} \quad \text{and} \quad \partial_{\underline{Z}^\dagger} \;=\; (\partial_{\underline{Z}})^\dagger = \sum_{j=1}^{n} \mathfrak{f}_j \, \partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and their complex conjugates $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ in the complex z_j–planes, $j = 1, \ldots, n$. The Hermitean vector variables and Dirac operators are isotropic, i.e.

$$(\underline{Z})^2 = (\underline{Z}^\dagger)^2 = 0 \quad \text{and} \quad (\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}^\dagger})^2 = 0$$

whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta_{2n} = 4(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}})$$

while also

$$\underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger\underline{Z} = |\underline{Z}|^2 = |\underline{Z}^\dagger|^2 = |\underline{X}|^2 = |\underline{X}|\,|^2$$

A continuously differentiable function g on an open region Ω of $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called (left) Hermitean monogenic (or h–monogenic) in Ω if and only if it simultaneously is $\partial_{\underline{X}}$– and $\partial_{\underline{X}|}$–monogenic in Ω, i.e. it satisfies in Ω the system

$$\partial_{\underline{X}}\, g = 0 = \partial_{\underline{X}|}\, g \quad \text{or the equivalent system} \quad \partial_{\underline{Z}}\, g = 0 = \partial_{\underline{Z}^\dagger}\, g$$

It remains to recall the group invariance underlying this system. To this end we consider the group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$, given by

$$\widetilde{\mathrm{U}}(n) = \{s \in \mathrm{Spin}(2n) \mid \exists\, \theta \geq 0 : \overline{s} I = \exp(-i\theta) I\}$$

its definition involving the self-adjoint primitive idempotent

$$I = I_1 \ldots I_n \tag{4}$$

with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}(1 - ie_j e_{n+j})$, $j = 1, \ldots, n$. It has been proved, see [9], that this group constitutes a realisation in the Clifford algebra of the unitary group $\mathrm{U}(n)$, and moreover, that its associated action leaves the Hermitean Dirac operators invariant. Less precisely, one thus says that these operators are invariant under the action of the unitary group, and so is the notion of h–monogenicity.

3 FUNDAMENTAL SOLUTIONS

The fundamental solutions of the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$, i.e. the orthogonal Cauchy kernels, are respectively given by

$$E\ (\underline{X}) = \frac{1}{a_{2n}}\frac{\overline{\underline{X}}}{|\underline{X}|^{2n}}, \quad E|(\underline{X}) = \frac{1}{a_{2n}}\frac{\overline{\underline{X}|}}{|\underline{X}|^{2n}}$$

where a_{2n} denotes the area of the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$. Explicitly, this means

$$\partial_{\underline{X}}E(\underline{X}) = \delta(\underline{X}), \quad \partial_{\underline{X}|}E|(\underline{X}) = \delta(\underline{X}|) = \delta(\underline{X})$$

In [6] we have also found

$$\begin{aligned}
\partial_{\underline{X}}E|(\underline{X}) &= -\frac{i}{n}(2\beta-n)\delta(\underline{X}) + 2n\frac{1}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^{2n+2}} - 2i(2\beta-n)\frac{1}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{|\underline{X}|^{2n}} \\
\partial_{\underline{X}|}E(\underline{X}) &= \frac{i}{n}(2\beta-n)\delta(\underline{X}) + 2n\frac{1}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{X}|\underline{X}}{|\underline{X}|^{2n+2}} + 2i(2\beta-n)\frac{1}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{|\underline{X}|^{2n}}
\end{aligned}$$

where β is the so–called spin Euler operator given by $\frac{1}{2}\sum_{j=1}^{n}(1-ie_je_{n+j})$ (see e.g. [9]) and Fp stands for the traditional "finite part" distribution. The Hermitean counterparts to the pair of fundamental solutions $(E, E|)$ are then given by

$$\mathcal{E}\ = -\left(E + i\,E|\right), \quad \mathcal{E}^\dagger = \quad \left(E - i\,E|\right)$$

or, explicitly:

$$\mathcal{E}\ (\underline{Z}) = \frac{2}{a_{2n}}\frac{\underline{Z}}{|\underline{Z}|^{2n}}, \quad \mathcal{E}^\dagger(\underline{Z}) = \frac{2}{a_{2n}}\frac{\underline{Z}^\dagger}{|\underline{Z}|^{2n}}$$

However, these are not the fundamental solutions to the respective Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, since the above results yield

$$\begin{aligned}
\partial_{\underline{Z}}\ \mathcal{E}(\underline{Z}) &= \frac{1}{n}\beta\delta(\underline{Z},\underline{Z}^\dagger) + \frac{2}{a_{2n}}\beta\mathrm{Fp}\,\frac{1}{r^{2n}} - \frac{2}{a_{2n}}n\mathrm{Fp}\,\frac{\underline{Z}^\dagger\underline{Z}}{r^{2n+2}} \\
\partial_{\underline{Z}^\dagger}\mathcal{E}(\underline{Z}) &= 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\underline{Z}}\ \mathcal{E}^\dagger(\underline{Z}) &= 0 \\
\partial_{\underline{Z}^\dagger}\mathcal{E}^\dagger(\underline{Z}) &= \frac{1}{n}(n-\beta)\delta(\underline{Z},\underline{Z}^\dagger) + \frac{2}{a_{2n}}(n-\beta)\mathrm{Fp}\,\frac{1}{r^{2n}} - \frac{2}{a_{2n}}n\mathrm{Fp}\,\frac{\underline{Z}\,\underline{Z}^\dagger}{r^{2n+2}}
\end{aligned}$$

Nevertheless, refined calculations on the blade level reveal

$$\partial_{\underline{Z}}\cdot\mathcal{E}(\underline{Z}) = \frac{1}{2}\delta(\underline{Z},\underline{Z}^\dagger) = \partial_{\underline{Z}^\dagger}\cdot\mathcal{E}^\dagger(\underline{Z})$$

whence $2\mathcal{E}(\underline{Z})$ may be interpreted as a fundamental solution of the operator $(\partial_{\underline{Z}}\cdot)$ and $2\mathcal{E}^\dagger(\underline{Z})$ as a fundamental solution of $(\partial_{\underline{Z}^\dagger}\cdot)$. Moreover, an important result was obtained in see [6], by considering the particular circulant (2×2) matrices

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

Indeed, it then holds that

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{\mathcal{E}}(\underline{Z}) = \boldsymbol{\delta}(\underline{Z})$$

meaning that $\boldsymbol{\mathcal{E}}$ may be considered as a fundamental solution of $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$. It is precisely this simple observation which has then lead us to the idea of a matrix approach to arrive at a Cauchy integral formula in the Hermitean setting. Also note, as another remarkable fact, that the Dirac matrix $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$ in some sense factorizes the Laplacian, since

$$4\,\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\left(\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right)^\dagger = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix}$$

Thus, in the same setting of circulant (2×2) matrices we associate, with continuously differentiable functions g_1 and g_2 defined in Ω and taking values in $\mathbb{C}_{2n}$, the matrix function

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

and we call $\boldsymbol{G}_2^1$ (left) $\boldsymbol{H}$–monogenic if and only if it satisfies the system $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_2^1 = \boldsymbol{O}$, where $\boldsymbol{O}$ denotes the matrix with zero entries. This system explicitly reads

$$\begin{cases} \partial_{\underline{Z}}\,[g_1] + \partial_{\underline{Z}^\dagger}[g_2] = 0 \\ \partial_{\underline{Z}^\dagger}[g_1] + \partial_{\underline{Z}}\,[g_2] = 0 \end{cases}$$

Choosing in particular $g_1 = g$ and $g_2 = 0$, it is clear that the $\boldsymbol{H}$–monogenicity of the corresponding matrix function

$$\boldsymbol{G}_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

is equivalent with the h–monogenicity of the function g, whence this specific matrix has turned out to be the key for the construction of a Cauchy integral formula for h–monogenic functions.

4 THE CAUCHY–BITSADSE AND TEODORESCU OPERATORS

As is well known, the fundamental solution $E(\underline{X})$ of the Euclidean Dirac operator is the key ingredient of the Cauchy integral formula for monogenic functions, of the Borel-Pompeiu formula for continuously differentiable functions and of the Teodorescu operator, i.e. the right inverse of the Dirac operator. Let us mention these fundamental results in Euclidean Clifford analysis for the sake of completeness.

Theorem 1 (Cauchy integral formula) *Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and let the function $f : \mathbb{R}^m \longrightarrow \mathbb{R}_{0,m}$ be (left-)monogenic in Ω and in $C_1(\overline{\Omega})$. Then for $\underline{X} \in \Omega$*

$$C_{\partial\Omega}[f](\underline{X}) = \int_{\partial\Omega} E(\underline{Y} - \underline{X})d\sigma_{\underline{Y}}f(\underline{Y}) = \int_{\partial\Omega} E(\underline{Y} - \underline{X})n_{\underline{Y}}f(\underline{Y})dS(\underline{Y}) = f(\underline{X}) \quad (5)$$

while for $\underline{X} \in \mathbb{R}^m \setminus \overline{\Omega}$ the integral is vanishing. Here, $n(\underline{Y})$ denotes the outward pointing unit normal vector at $\underline{Y}$.

The operator $C_{\partial\Omega}$ arising in the above theorem is usually called the Cauchy–Bitsadse operator. Note that explicitly

$$d\sigma_{\underline{Y}} = \sum_{j=1}^{n} e_j\,(-1)^{j-1}\,\widetilde{\widetilde{dx_j}} + \sum_{j=1}^{n} e_{n+j}\,(-1)^{n+j-1}\,\widetilde{\widetilde{dy_j}}$$

where

$$\begin{aligned}\widetilde{\widetilde{dx_j}} &= dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n\\ \widetilde{\widetilde{dy_j}} &= dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n\end{aligned}$$

in the original consecutive ordering of the variables $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Theorem 2 (Borel–Pompeiu formula) *Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and let the function $g : \mathbb{R}^m \longrightarrow \mathbb{R}_{0,m}$ be in $C_1(\overline{\Omega})$. Then for $\underline{X} \in \Omega$*

$$\int_{\partial\Omega} E(\underline{Y}-\underline{X})d\sigma_{\underline{Y}}g(\underline{Y}) - \int_{\Omega} E(\underline{Y}-\underline{X})\partial_{\underline{Y}}g(\underline{Y})dV(\underline{Y}) = g(\underline{X}) \tag{6}$$

while for $\underline{X} \in \mathbb{R}^m \setminus \overline{\Omega}$ the left hand side vanishes.

Note that the Cauchy integral formula (5) is a special case of the Borel–Pompeiu formula (6). Here we explicitly have $dV(\underline{Y}) = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$.

Theorem 3 (Teodorescu inversion) *Let Ω be a bounded domain in $\mathbb{R}^m$ and let the function $u : \mathbb{R}^m \longrightarrow \mathbb{R}_{0,m}$ be in $C_1(\Omega)$. Then*

$$T_\Omega[u](\underline{X}) = -\int_{\Omega} E(\underline{Y}-\underline{X})u(\underline{Y})dV(\underline{Y}) \tag{7}$$

belongs to $C_1(\Omega)$ and satisfies in Ω

$$\partial_{\underline{X}} T_\Omega[u](\underline{X}) = u(\underline{X})$$

The operator T_Ω is usually called the Teodorescu operator; it is the right inverse of $\partial_{\underline{X}}$ in Ω.

The traditional Koppelman formula of several complex variables expresses the value of a continuously differentiable function or differential form in an interior point of a bounded domain by means of a sum of an integral over the boundary, an integral over the domain itself and an integral over the domain subject to the action of the "d–bar" operator $\overline{\partial}$. When analyzing its proof, e.g. in [15], it becomnes clear that the Koppelman formula originates as a combination of the Borel–Pompeiu formula and a kind of Teodorescu inverse of $\overline{\partial}$, the last one up to the principal value of a singular integral. Here, by combining formulae (6) and (7) one obtains for $\underline{X} \in \Omega$

$$\int_{\partial\Omega} E(\underline{Y}-\underline{X})d\sigma_{\underline{Y}}g(\underline{Y}) - \int_{\Omega} E(\underline{Y}-\underline{X})\partial_{\underline{Y}}g(\underline{Y})dV(\underline{Y}) - \partial_{\underline{X}}\int_{\Omega} E(\underline{Y}-\underline{X})g(\underline{Y})dV(\underline{Y}) = 2g(\underline{X}) \tag{8}$$

which is to be seen as a Clifford–Koppelman formula.

Let us now reconsider these results in Hermitean Clifford analysis, so take a bounded domain $\Omega \subset \mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$. As for the vector variable $\underline{X}$ and for the Dirac operator $\partial_{\underline{X}}$, a twisted version of the above operators and theorems may be considered as well, using the fundamental solution $E|$ of $\partial_{\underline{X}|}$. The following results hold.

Theorem 4 (twisted Cauchy integral formula) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$ and let the function $f : \mathbb{R}^{2n} \longrightarrow \mathbb{C}_{2n}$ be (left-)monogenic in Ω w.r.t. $\partial_{\underline{X}|}$ and in $C_1(\overline{\Omega})$. Then for $\underline{X} \in \Omega$*

$$C|_{\partial\Omega}[f](\underline{X}) = \int_{\partial\Omega} E|(\underline{Y} - \underline{X}) d\sigma_{\underline{Y}|} f(\underline{Y}) = f(\underline{X}) \tag{9}$$

In the above theorem, we have used a twisted surface element

$$d\sigma_{\underline{Y}|} = J[d\sigma_{\underline{Y}}] = \sum_{j=1}^{n} e_j \, (-1)^{n+j-1} \, \widetilde{\widehat{dy_j}} - \sum_{j=1}^{n} e_{n+j} \, (-1)^{j-1} \, \widetilde{\widehat{dx_j}}$$

Theorem 5 (twisted Teodorescu inversion) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ and let the function $u : \mathbb{R}^{2n} \longrightarrow \mathbb{C}_{2n}$ be in $C_1(\Omega)$. Then*

$$T|_{\Omega}[u](\underline{X}) = -\int_{\Omega} E|(\underline{Y} - \underline{X}) u(\underline{Y}) dV(\underline{Y}) \tag{10}$$

belongs to $C_1(\Omega)$ and satisfies in Ω

$$\partial_{\underline{X}|} T|_{\Omega}[u](\underline{X}) = u(\underline{X})$$

Theorem 6 (twisted Borel–Pompeiu formula) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$ and let the function $g : \mathbb{R}^{2n} \longrightarrow \mathbb{C}_{2n}$ be in $C_1(\overline{\Omega})$. Then for $\underline{X} \in \Omega$*

$$C|_{\partial\Omega}[g](\underline{X}) + T|_{\Omega}[\partial_{\underline{X}|} g](\underline{X}) = g(\underline{X}) \tag{11}$$

We will first make complex linear combinations of the Teodorescu operators:

$$\begin{aligned}
\mathcal{T}_{\Omega}^{(1)}[(\cdot)](\underline{Z}) &= (-1)^{\frac{n(n+1)}{2}} (2i)^n (-T_{\Omega} - iT|_{\Omega}) = -\int_{\Omega} \mathcal{E}(\underline{W} - \underline{Z})(\cdot)(\underline{W}) \, dW \\
\mathcal{T}_{\Omega}^{(2)}[(\cdot)](\underline{Z}) &= (-1)^{\frac{n(n+1)}{2}} (2i)^n (T_{\Omega} - iT|_{\Omega}) = -\int_{\Omega} \mathcal{E}^{\dagger}(\underline{W} - \underline{Z})(\cdot)(\underline{W}) \, dW
\end{aligned}$$

where we have used the notation

$$dW = (dw_1 \wedge dw_1^c) \wedge (dw_2 \wedge dw_2^c) \wedge \cdots \wedge (dw_n \wedge dw_n^c) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \, dV(\underline{Y})$$

Using these definitions, we arrive at a Hermitean version of the Teodorescu inversion.

Theorem 7 (Hermitean Teodorescu inversion) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$, and let $u \in C_1(\Omega)$. Then $\mathcal{T}_{\Omega}^{(1)}[u]$ and $\mathcal{T}_{\Omega}^{(2)}[u]$ are in $C_1(\Omega)$ and satisfy in Ω*

(i) $\partial_{\underline{Z}} \mathcal{T}_{\Omega}^{(1)}[u] + \partial_{\underline{Z}^{\dagger}} \mathcal{T}_{\Omega}^{(2)}[u] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \, u$;

(ii) $\partial_{\underline{Z}^{\dagger}} \mathcal{T}_{\Omega}^{(1)}[u] + \partial_{\underline{Z}} \mathcal{T}_{\Omega}^{(2)}[u] = 0$.

Note that the results of the foregoing theorem may be written in matrix form as

$$\left(\begin{array}{cc} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{array} \right) \left(\begin{array}{cc} \mathcal{T}_\Omega^{(1)}[u] & \mathcal{T}_\Omega^{(2)}[u] \\ \mathcal{T}_\Omega^{(2)}[u] & \mathcal{T}_\Omega^{(1)}[u] \end{array} \right) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\begin{array}{cc} u & 0 \\ 0 & u \end{array} \right)$$

or still, using the explicit formulae and the matrix definitions of the foregoing section

$$-\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \int_\Omega \boldsymbol{\mathcal{E}}(\underline{W}-\underline{Z}) \left(\begin{array}{cc} u(\underline{W}) & 0 \\ 0 & u(\underline{W}) \end{array} \right) dW = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\begin{array}{cc} u(\underline{W}) & 0 \\ 0 & u(\underline{W}) \end{array} \right)$$

Refined calculations at the blade level yield

$$\begin{aligned}
\partial_{\underline{Z}} \cdot \mathcal{T}_\Omega^{(1)}[u] &= \frac{1}{2}(-1)^{\frac{n(n+1)}{2}}(2i)^n u(\underline{Z}) \\
\partial_{\underline{Z}} \wedge \mathcal{T}_\Omega^{(1)}[u] &= \frac{1}{n}(-1)^{\frac{n(n+1)}{2}}(2i)^n \left(\sum_{j=1}^n \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \right) u(\underline{Z}) + \mathrm{Pv} \int_\Omega \partial_{\underline{Z}} \mathcal{E}(\underline{Z}-\underline{W}) u(\underline{W})\, dW \\
\partial_{\underline{Z}^\dagger} \cdot \mathcal{T}_\Omega^{(1)}[u] &= 0 \\
\partial_{\underline{Z}^\dagger} \wedge \mathcal{T}_\Omega^{(1)}[u] &= 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\underline{Z}} \cdot \mathcal{T}_\Omega^{(2)}[u] &= 0 \\
\partial_{\underline{Z}} \wedge \mathcal{T}_\Omega^{(2)}[u] &= 0 \\
\partial_{\underline{Z}^\dagger} \cdot \mathcal{T}_\Omega^{(2)}[u] &= \frac{1}{2}(-1)^{\frac{n(n+1)}{2}}(2i)^n u(\underline{Z}) \\
\partial_{\underline{Z}^\dagger} \wedge \mathcal{T}_\Omega^{(2)}[u] &= \frac{1}{n}(-1)^{\frac{n(n+1)}{2}}(2i)^n \left(\sum_{j=1}^n \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger \right) u(\underline{Z}) + \mathrm{Pv} \int_\Omega \partial_{\underline{Z}^\dagger} \mathcal{E}^\dagger(\underline{Z}-\underline{W}) u(\underline{W})\, dW
\end{aligned}$$

Returning to the beginning of this section, we will now also make complex linear combinations of the Cauchy–Bitsadse operators mentioned there:

$$\begin{aligned}
\mathcal{C}_{\partial\Omega}^{(1)}[(\cdot)](\underline{Z}) &= (-1)^{\frac{n(n+1)}{2}}(2i)^n (C_{\partial\Omega} + C|_{\partial\Omega}) \\
&= \int_{\partial\Omega} (\mathcal{E}(\underline{Z}-\underline{W}) d\sigma_{\underline{W}} - \mathcal{E}^\dagger(\underline{W}-\underline{Z}) d\sigma_{\underline{W}}^\dagger)(\cdot)(\underline{W}) \\
\mathcal{C}_{\partial\Omega}^{(2)}[(\cdot)](\underline{Z}) &= (-1)^{\frac{n(n+1)}{2}}(2i)^n (C_{\partial\Omega} - C|_{\partial\Omega}) \\
&= \int_{\partial\Omega} (\mathcal{E}^\dagger(\underline{Z}-\underline{W}) d\sigma_{\underline{W}} - \mathcal{E}(\underline{W}-\underline{Z}) d\sigma_{\underline{W}}^\dagger)(\cdot)(\underline{W})
\end{aligned}$$

where we have used the notations

$$\begin{aligned}
d\sigma_{\underline{W}} &= = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n (d\sigma_{\underline{Y}} - i d\sigma_{\underline{Y}|}) \ = \ \sum_{j=1}^n \mathfrak{f}_j^\dagger \widehat{dw_j} \\
d\sigma_{\underline{W}}^\dagger &= = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n (d\sigma_{\underline{Y}} - i d\sigma_{\underline{Y}|}) \ = \ \sum_{j=1}^n \mathfrak{f}_j^\dagger \widehat{dw_j}
\end{aligned}$$

Using these definitions, we will now establish a Hermitean version of the Cauchy integral formula and of the Borel–Pompeiu formula.

Theorem 8 (Hermitean Borel–Pompeiu formula) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$ and let the function $g : \mathbb{R}^{2n} \longrightarrow \mathbb{C}_{2n}$ be in $C_1(\overline{\Omega})$. Then, for $\underline{X} \in \Omega$*

(i) $\mathcal{C}^{(1)}_{\partial\Omega}[g](\underline{Z}) + \mathcal{T}^{(1)}_{\Omega}[\partial_{\underline{W}} g](\underline{Z}) + \mathcal{T}^{(2)}_{\Omega}[\partial^{\dagger}_{\underline{W}} g](\underline{Z}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n\, g(\underline{Z})$;

(ii) $\mathcal{C}^{(2)}_{\partial\Omega}[g](\underline{Z}) + \mathcal{T}^{(1)}_{\Omega}[\partial^{\dagger}_{\underline{W}} g](\underline{Z}) + \mathcal{T}^{(2)}_{\Omega}[\partial_{\underline{W}} g](\underline{Z}) = 0.$

First note that this result may also be expressed in circulant matrix form, but even more important, that it reduces to a Hermitean Cauchy formula for functions g which are Hermitean monogenic.

Theorem 9 (Hermitean Cauchy integral formula) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$ and let the function $g : \mathbb{R}^{2n} \longrightarrow \mathbb{C}_{2n}$ be in $C_1(\overline{\Omega})$. If the function g moreover is h–monogenic in Ω, then, for $\underline{X} \in \Omega$*

$$\int_{\partial\Omega} \boldsymbol{\mathcal{E}}(\underline{W} - \underline{Z})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^{\dagger})}\, \boldsymbol{G}_0(\underline{W}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \boldsymbol{G}_0(\underline{Z})$$

where

$$\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^{\dagger})} = \begin{pmatrix} d\sigma_{\underline{W}} & -d\sigma_{\underline{W}^{\dagger}} \\ -d\sigma_{\underline{W}^{\dagger}} & d\sigma_{\underline{W}} \end{pmatrix}$$

According to the remark made in previous section, the previous theorem may indeed be considered as a Hermitean Cauchy integral formula for the h–monogenic function g; therefore the matrix function $\boldsymbol{\mathcal{E}}$ appearing in this formula is called the Hermitean Cauchy kernel.

The Cauchy integral is a well–known integral operator applying to functions defined on $\partial\Omega$. It has been thoroughly studied in the framework of Euclidean Clifford analysis; in particular it has lead to the definition of Clifford–Hardy spaces and of a multidimensional Clifford vector valued Hilbert transform, when considering its non–tangential boundary limits in L_2–sense in the interior or exterior of the domain of interest (see [12, 10]). It is clear that, by means of the matricial Hermitean Cauchy kernel defined in this section, also a Hermitean Cauchy integral may be defined; the study of its boundary limits, leading to Hermitean Clifford–Hardy spaces and to a Hermitean Hilbert transform, is the subject of the paper [5].

Further refined calculations then also reveal Hermitean Clifford–Koppelman formulae.

Theorem 10 *Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$ and let the function $f : \mathbb{R}^{2n} \longrightarrow \mathbb{C}_{2n}$ be in $C_1(\overline{\Omega})$. Then, for $\underline{X} \in \Omega$*

$$\begin{aligned}
&(-1)^{\frac{n(n+1)}{2}} (2i)^n\, f(\underline{Z}, \underline{Z}^{\dagger}) \\
&= \int_{\partial\Omega} \mathcal{E}(\underline{W} - \underline{Z}) d\sigma_{\underline{W}} f(\underline{W}) - \int_{\Omega} \mathcal{E}(\underline{W} - \underline{Z}) \partial_{\underline{W}} f dW - \partial_{\underline{Z}} \int_{\Omega} \mathcal{E}(\underline{W} - \underline{Z}) f(\underline{W}) dW \\
&= -\int_{\partial\Omega} \mathcal{E}^{\dagger}(\underline{W} - \underline{Z}) d\sigma^{\dagger}_{\underline{W}} f(\underline{W}) - \int_{\Omega} \mathcal{E}^{\dagger}(\underline{W} - \underline{Z}) \partial^{\dagger}_{\underline{W}} f dW - \partial_{\underline{Z}^{\dagger}} \int_{\Omega} \mathcal{E}^{\dagger}(\underline{W} - \underline{Z}) f(\underline{W}) dW
\end{aligned}$$

5 SPECIAL CASES

As a first special case we consider a scalar valued function $u \in C_1(\overline{\Omega})$ with which we form the spinor valued function $f = u\mathfrak{f}^{\dagger}_1 \mathfrak{f}^{\dagger}_2 \ldots \mathfrak{f}^{\dagger}_n I$, with I the idempotent (4). For the function f it holds that

$$\partial_{\underline{Z}} f = 0 \quad \text{and} \quad \partial_{\underline{Z}^\dagger} f = \sum_{k=1}^{n} \mathfrak{f}_k (\partial_{z_k} u) \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger I \tag{12}$$

The Hermitean Borel–Pompeiu formulae then become, for $\underline{X} \in \Omega$,

$$(-1)^{\frac{n(n+1)}{2}} u(\underline{Z}, \underline{Z}^\dagger) = -\frac{2}{a_{2n}} \int_{\partial\Omega} \sum_{j=1}^{n} \frac{w_j^c - z_j^c}{\rho^{2n}} \widehat{dw_j^c}\, u - \frac{2}{a_{2n}} \int_{\Omega} \sum_{j=1}^{n} \frac{w_j^c - z_j^c}{\rho^{2n}} (\partial_{w_j^c} u)\, dW$$

and, still with $\underline{X} \in \Omega$, for $j \neq k$,

$$\int_{\partial\Omega} \frac{u}{\rho^{2n}} \left((w_j - z_j) \widehat{dw_k^c} - (w_k - z_k) \widehat{dw_j^c} \right) + \int_{\Omega} \frac{(w_j - z_j)(\partial_{w_k^c} u) - (w_k - z_k)(\partial_{w_j^c} u)}{\rho^{2n}}\, dW = 0$$

where $\rho = |\underline{Z} - \underline{W}|$.

In the first formula we recognize a well-known formula from several complex variable theory, the so–called Bochner–Martinelli formula for smooth functions. When f is h–monogenic, which, on account of (12) is seen to be equivalent to the holomorphy of u, it reduces to

$$(-1)^{\frac{n(n+1)}{2}} u(\underline{Z}, \underline{Z}^\dagger) = -\frac{2}{a_{2n}} \int_{\partial\Omega} \sum_{j=1}^{n} \frac{w_j^c - z_j^c}{\rho^{2n}} \widehat{dw_j^c}\, u(\underline{W}, \underline{W}^\dagger)$$

which is the Bochner–Martinelli formula (2) for holomorphic functions, mentioned in the introduction, when taking into account the appropriate reordering of the involved differential forms.

A second special case, leading to very similar calculations, results and conclusions, occurs when considering a scalar valued function $u \in C_1(\overline{\Omega})$ and forming with it the spinor valued function $g = u \mathfrak{f}_1 \mathfrak{f}_2 \ldots \mathfrak{f}_n K$, where K is another idempotent, given by

$$K = \mathfrak{f}_1^\dagger \mathfrak{f}_1 \mathfrak{f}_2^\dagger \mathfrak{f}_2 \ldots \mathfrak{f}_n^\dagger \mathfrak{f}_n$$

For the function g it holds that

$$\partial_{\underline{Z}^\dagger} g = 0 \quad \text{and} \quad \partial_{\underline{Z}} g = \sum_{k=1}^{n} \mathfrak{f}_k^\dagger (\partial_{z_k^c} u) \mathfrak{f}_1 \mathfrak{f}_2 \ldots \mathfrak{f}_n K$$

whence its h–monogenicity now is seen to be equivalent to the antiholomorphy of u.

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